Consistent local approximation in continuous time

Tamás K. Papp*

April 19, 2017

Abstract

Analysis of the approximation method of Den Haan, Kobielarz, and Rendahl (2015) and Levintal (2016), applied to the deterministic Ramsey model in continuous time. I show that while the method is easy to set up, solving the nonlinear system requires nontrivial methods for even a simple system, and once solved, the resulting residuals of the Euler equation are large compared to collocation methods, but still small enough in absolute magitude to make the model useful in practice, especially for making an initial guess about functional forms in collocations methods.

See the repository at https://github.com/tpapp/consistent-local-continuous for the accompanying code in Julia.

Introduction. Continuous time models have been used extensively in macroeconomics, finance, and other fields[1] However, while in finance it is common practice to solve continuous models numerically,[2] in macroeconomics practical applications of numerical methods usually focus on discrete time problems, despite the fact that some commonly used textbooks on numerical methods address continuous-time methods.[3]

While one of the reasons for this may be unfamiliarity with continuous-time building blocks such as Itô and Lévy processes,[4] the other difficulty is finding a solution to the resulting functional equations. When using projection methods, functional forms that describe optimal choices and values are approximated using function families with finitely many parameters. Since in continuous time there is no general contraction mapping equivalent to value iteration in discrete time, usually the only option is to solve these problems using general nonlinear solvers, which are not guaranteed to converge, especially when the starting point is not close enough to the solution.[5]

Den Haan, Kobielarz, and Rendahl (2015) propose a method for finding approximate solutions to discrete time systems described by a functional equation, by approximating policy functions with a linear (or similarly simple) form and imposing consistency between present and future policies around a particular state. Levintal (2016) proposed a similar method, while a very similar approach can be found in Krusell, Kuruşçu, and Smith (2002, Appendix B). In this short note I show that this method can be applied to continuous time systems, and examine its practicality and accuracy in the context of one of the simplest continuous time models, the deterministic Ramsey model. The advantage of

*E-mail: tkpapp@gmail.com. This note was prepared for the 2017 Festschrift in honor of Júlia Király, who I was fortunate to have as a teacher. I would like to thank Pontus Rendahl and Anna Naszódi for comments.

[1] References are too numerous to list here. See Acemoglu (2008) for a textbook based almost entirely on continuous time models, Dixit, Pindyck, and Davis (1996) for applications to inverstment. Applications in portfolio choice and finance started with Merton (1971).

[2] See Hull (2009) for commonly used methods

[3] Eg Judd (1998) and Miranda and Fackler (2002). Recent exceptions include Kaplan, Moll, and Violante (2016) and related papers.

[4] For an introduction, see Øksendal and Sulem (2005).

[5] Kushner and Dupuis (2013) presents an alternative method, using discrete-time approximations of continuous processes.

this model is that it is relatively well-studied,[6] and despite its simplicity, it is sufficient for making the following points about the proposed method in continuous time:

1. The method is fast, but the resulting nonlinear system may have multiple roots even in a simple setting, so that it must be solved with care, preferably from a good initial guess. In this note I used continuation methods, starting from around the steady state.

2. The method less accurate compared to collocation methods: Euler equation residuals deteriorate very quickly with distance form the steady state. For example, at half the steady state capital, the relative residual was 10^{-2}, while a simple collocation method with 10 Chebyshev polynomials can easily achieve 10^{-4}. However, it is still more than accurate enough as a starting point for a nonlinear solver, so it can be used as an initial guess about the functional form for collocation methods.

3. The method is good at capturing the general shape of the policy function even near singularities, where collocation methods usually break down without transformations.

Setup. Consider a deterministic Ramsey model in continuous time, with CRRA utility function (IES θ), discount rate ρ, production function

$$F(k) = Ak^{\alpha} - \delta k$$

where k_t is the capital stock, which develops according to the capital accumulation equation

$$\dot{k}_t = F(k_t) - c_t \tag{1}$$

where c_t is consumption. The most frequently used form of the Euler equation, usually obtained from the Hamiltonian, is[7]

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\big(F'(k_t) - \rho\big) \tag{2}$$

First, I rewrite this into a recursive form. We are solving for the policy function $c(k)$, and thus

$$\dot{c} = c'(k)\dot{k} = c'(k)\big(F(k) - c(k)\big)$$

where I have used (1) and dropped time indices. Plugging into (2), we obtain

$$\frac{c'(k)}{c(k)}\big(F(k) - c(k)\big) = \frac{1}{\theta}\big(F'(k) - \rho\big) \tag{3}$$

We cast this into the form

$$c'(k)\big(F(k) - c(k)\big) = \frac{1}{\theta}\big(F'(k) - \rho\big)c(k) \tag{4}$$

which should be easier to manipulate. We are looking for the solution $c(k)$ to (4).

Methodology. The key to the method outlined is the following:

1. fix k,

2. assume a functional form for $c(k)$ around this point,

3. solve (4) by imposing that this form holds "locally", as described below.

[6] Aruoba, Fernandez-Villaverde, and Rubio-Ramirez (2006) compare a variety of numerical methods for the discrete-time version of this model with a stochastic productivity.

[7] Eg see Acemoglu (2008, Chapter 8).

For simplicity, we choose a linear form[8]

$$c(\tilde{k};k) = c_0(k) + c_1(k)(\tilde{k} - k) \tag{5}$$

First, from (4) and (5) we obtain

$$c_1(k)\big(F(k) - c_0(k)\big) = \frac{1}{\theta}\big(F'(k) - \rho\big)c_0(k) \tag{6}$$

Implicit differentiation by k yields

$$c_1'(k)\big(F(k) - c_0(k)\big) + c_1(k)\big(F'(k) - c_0'(k)\big) = \frac{1}{\theta}\Big[F''(k)c_0(k) + \big(F'(k) - \rho\big)c_0'(k)\Big] \tag{7}$$

The method makes two assumptions. First, we impose that the approximation is *valid locally* around k:

$$c_0'(k) = c_1(k) \tag{8}$$

This would hold if the $c(\tilde{k};k)$ was tangent to the approximated policy function $c_0(k)$. Also, we impose that for a small change in k, there is *no first-order change in the approximating slope* $c_1(k)$:

$$c_1'(k) = 0 \tag{9}$$

This is a crucial assumption, as it basically imposes no curvature. The accuracy of the method will be determined by how realistic these assumptions are. Using (8) and (9), (7) becomes

$$c_1(k)\big(F'(k) - c_1(k)\big) = \frac{1}{\theta}\Big[F''(k)c_0(k) + \big(F'(k) - \rho\big)c_1(k)\Big] \tag{10}$$

For each k, we solve the system of (5) and (10). Note that in contrast to perturbation methods, (10) still takes the local curvature of F into account, contributing to the accuracy of the method. When comparing to projection methods, it is important to note that the consistent local approximation does not rely on knowing the solution at other gridpoints, which is advantageous for problems which suffer from the curse of dimensionality.

Numerical methods. Note that the system (6) and (10) is quadratic. Using random starting points and quasi-Newton methods,[9] I generally found *three* solutions: the "right" one ($c_1(k), c_0(k) > 0$), a trivial one $c_1(k) = c_0(k) = 0$, and an an economically nonsensical one (with $c_1(k) < 0$). Consequently, reformulating the problem and investing time in finding good initial guesses is beneficial. Using (3) rules out the zero solution. Also note that since the steady state capital can be obtained as

$$k_s = \left(\frac{\delta + \rho}{A\alpha}\right)^{1/(\alpha-1)}$$

and $c_s = c_0(k) = F(k_s)$, an exact solution can be easily solved for at the steady state. However, even with seemingly sensible initial guesses,[10] the quasi-Newton solver frequently converged the nonsensical root, so I used the following simple continuation method:

1. for a k near k_s, use initial guesses $c_0(k) = c_s$, and $c_1(k)$ from (10),
2. for other k, find a nearby $\tilde{k}$ for which we have solved the problem, and use $c_0(\tilde{k}), c_1(\tilde{k})$.

[8] Den Haan, Kobielarz, and Rendahl (2015) argue that higher-order approximations would give even better accuracy. However, since the most difficult part is solving the resulting nonlinear system, there is a trade-off between accuracy and programmer time. That said, exploring Padé approximations would be interesting.

[9] Unless stated otherwise, in this note I use quasi-Newton methods with automatic differentiation (Revels, Lubin, and Papamarkou 2016).

[10] Such as (11) which works very well for collocation.

For comparison, I also solve (3) using a collocation method with 10 Chebyshev polynomials, using a quasi-Newton solver with linesearch and the initial guess

$$c(k) = c(k_s)\frac{k}{k_s} \tag{11}$$

This is very accurate and will be useful for comparison. I refer to the collocation solution as the "exact" solution $\hat{c}(k)$, since the approximation error has no practical relevance.

Solution and discussion. I solve on the range $k \in (0.2k_s, 2k_s)$, with parameters $A = 1$, $\alpha = 0.3$, $\rho = 0.02$, $\delta = 0.05$, $\theta = 2$. This solution is referred to in the graphs as "collocation", while the method discussed in this paper is labeled as "consistent local".

Figure 1 shows the policy function. The deviation from the exact solution is small but visible, with the exact solution method overpredicting consumption both above and below the steady state. More importantly, as can be seen in Figure 2 which shows the same comparison for $c_1(k)$ and and $\hat{c}'(k)$, the consistent local method overpredicts $c'(k)$ above and underpredicts it below the steady state, respectively. Also, note that (9) is violated since $c_1(k)$ appears to have significant curvature.

Figure 3 checks (8) by plotting $c_1(0)$ and $c_0'(k)$, calculating the latter from a local quadratic curve. While (8) holds at the steady state by construction, it is violated significantly the further we move from the steady state: as k approaches 0, $c_0'(k) \approx 2c_1(k)$. If we are aiming for a globally accurate solution, (8) turns out to be an unreasonable assumption. Figure 4 shows $c_0(k)$, but with tangents drawn using $c_0(k)$ and $c_1(k)$. This is another graphical illustration of the volation of (8).

Figures 5 and 6 show the residual of (3) for the consistent local and collocation methods, respectively. Graphing this residual is a standard procedure of evaluating collocation methods (Boyd 2001; Judd 1998), since the residual can be considered as a unitless prediction error. Note that because of the errors discussed above, the consistent local method displays relatively low accuracy when compared to collocation methods, however, the residual is still small for practical purposes.

One remarkable feature of the consistent local method is that because of its locality, it does not break down as $k \to 0$, in contrast to collocation methods, which require transformations or more gridpoints in areas of high curvature. This again makes it ideal for initial guesses and exploratory analysis.

Conclusion. I examined the accuracy of the method proposed by Den Haan, Kobielarz, and Rendahl (2015) in the context of a simple continuous time Ramsey model, and found that it is much less accurate than collocation methods. Note, however, that the method is still reasonably accurate, and since it can be implemented very easily, it may be a good first pass approach to study the solution of functional equations. In particular, it can be used for exploratory analysis, or to provide initial guesses for non-linear solvers in collocation methods, and since it can be evaluated at any points, it can also be used to check the assumptions about functional forms that underlie more complex methods, such as sparse grid approximations (Judd et al. 2014). The only part which requires more care is solving the nonlinear system. With these in mind, the method may be a useful addition to our toolkit for solving functional equations in economics.

References

Acemoglu, Daron (2008). *Introduction to modern economic growth.* Princeton University Press.

Aruoba, S Borağan, Jesus Fernandez-Villaverde, and Juan F Rubio-Ramirez (2006). "Comparing solution methods for dynamic equilibrium economies". In: *Journal of Economic dynamics and Control* 30.12, pp. 2477–2508.

Bezanson, Jeff et al. (2017). "Julia: A Fresh Approach to Numerical Computing". In: *SIAM Review* 59.1, pp. 65–98. DOI: 10.1137/141000671. eprint: http://dx.doi.org/10.1137/141000671. URL: http://dx.doi.org/10.1137/141000671.

Boyd, John P (2001). *Chebyshev and Fourier spectral methods.* Courier Corporation.

Den Haan, Wouter J, Michal L Kobielarz, and Pontus Rendahl (2015). *Exact present solution with consistent future approximation: A gridless algorithm to solve stochastic dynamic models.* Tech. rep.

Dixit, Avinash K, Robert S Pindyck, and Graham A Davis (1996). "Investment under uncertainty". In: *Resources Policy* 22.3, p. 217.

Hull, John (2009). *Options, Futures and Other Derivatives.* Pearson Education.

Judd, Kenneth L (1998). *Numerical methods in economics.* MIT Press.

Judd, Kenneth L et al. (2014). "Smolyak method for solving dynamic economic models: Lagrange interpolation, anisotropic grid and adaptive domain". In: *Journal of Economic Dynamics and Control* 44, pp. 92–123.

Kaplan, Greg, Benjamin Moll, and Giovanni L Violante (2016). *Monetary policy according to HANK.* Tech. rep. National Bureau of Economic Research.

Krusell, Per, Burhanettin Kuruşçu, and Anthony A Smith (2002). "Equilibrium welfare and government policy with quasi-geometric discounting". In: *Journal of Economic Theory* 105.1, pp. 42–72.

Kushner, Harold and Paul G Dupuis (2013). *Numerical methods for stochastic control problems in continuous time.* Vol. 24. Springer Science & Business Media.

Levintal, Oren (2016). "Taylor Projection: A New Solution Method for Dynamic General Equilibrium Models". In: *Browser Download This Paper.*

Merton, Robert C (1971). "Optimum consumption and portfolio rules in a continuous-time model". In: *Journal of economic theory* 3.4, pp. 373–413.

Miranda, Mario J and Paul L Fackler (2002). *Applied computational economics and finance.* MIT Press.

Øksendal, Bernt Karsten and Agnes Sulem (2005). *Applied stochastic control of jump diffusions.* Vol. 498. Springer.

Revels, J., M. Lubin, and T. Papamarkou (2016). "Forward-Mode Automatic Differentiation in Julia". In: *arXiv:1607.07892 [cs.MS].* URL: https://arxiv.org/abs/1607.07892.

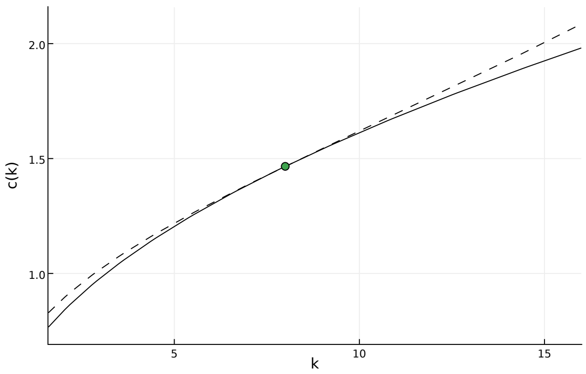

Figure 1: Policy functions (consistent local: dashed, collocation: solid, steady state: dot).

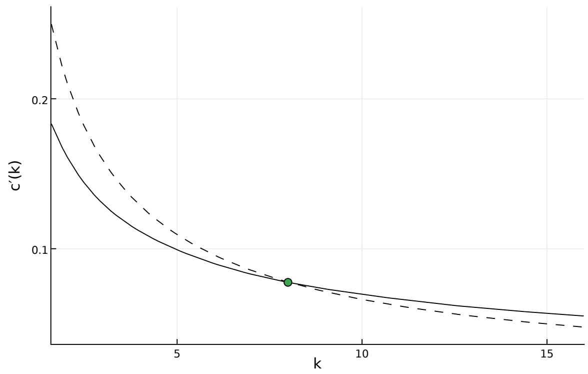

Figure 2: $c_1(k)$ for consistent local approximation (dashed), $c'(k)$ for collocation (solid), steady state (dot).

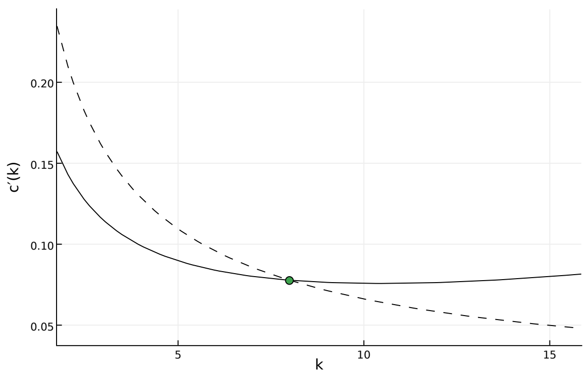

Figure 3: $c_1(k)$ (solid) and $\partial c_0(k)/\partial k$ (dashed), the latter calculated from a local quadratic approximation.

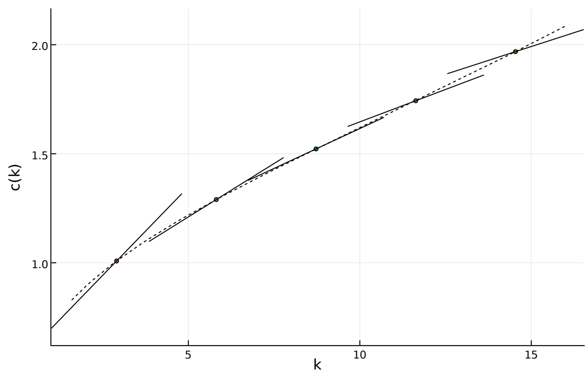

Figure 4: Policy function $c_0(k)$, with tangents drawn according to $c_1(k)$.

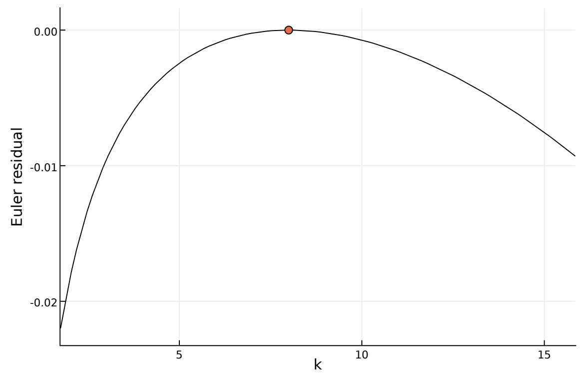

Figure 5: Residual of equation (3) for the consistent local method.

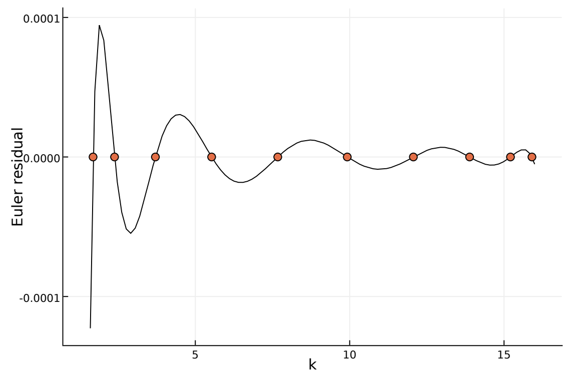

Figure 6: Residual of equation (3) for the collocation method, with collocation nodes.